S.I.Adian

On Novikov–Adian's method in combinatorial group theory.

Abstract.

In 1902, W.Burnside [1] formulated the following problem:

> *For* $r, n \geq 2$ *consider the group*
>
> $$B(r,n) = \langle a_1, \ldots, a_r \mid x^n = 1 \rangle \qquad (1)$$
>
> *presented by the identity relation* $x^n = 1$ *in the generators* $a_1, \ldots, a_r$*. Is the group* $B(r,n)$ *always finite?*

The groups $B(r,n)$ are called *free Burnside groups of exponent* n.

The finiteness of the groups $B(r,n)$ for any $r > 1$ was proved only for $n \leq 3$ (W. Burnside, 1902), $n = 4$ (I. Sanov, 1940) and $n = 6$ (M. Hall, 1958). The finiteness of $B(r,5)$ for $r > 1$ still is an open question.

A negative solution of the Burnside problem on periodic groups was given by P.S. Novikov and S.I. Adian in the joint papers [2]. It was proved that the free periodic groups $B(r,n)$ for any $r > 1$ generators and odd $n \geq 4381$ are infinite. Obviously, from $B(r,n) = \infty$ it follows $B(r,kn) = \infty$ for any $k > 1$.

In the monograph [3] devoted to the history of combinatorial group theory, Prof. Wilhelm Magnus characterized the Burnside problem as follows (see page 154, line 21):

> *Very much like Fermat's Last Theorem in number theory, Burnside's problem has acted as a catalyst for research in group theory. The fascination exerted by a problem with an extremely simple formulation which then turns out to be extremely difficult has something irresistible about it to the mind of the mathematician.*

Usually, a solution of very old famous problem (for instance the Fermat's problem or the Poincare's conjecture) is based on some essential

preceding investigations of another authors. But for the negative solution of the Burnside problem in [2] the authors used nothing special that could be unknown in 1902 and was proved later.

For a solution of the Burnside problem in [2] a new method was created. It is based on a classification of periodic words and an appropriate system of defining relations for the group $B(m, n)$ obtained by simultaneous induction on a natural parameter α called *rank*.

In the book [4] the author presented an improved version of the new theory for odd exponents $n \geq 665$ and proved some other applications of the method. The improvement in [4] allowed to use the theory as a new powerful method to construct various groups with given properties. These groups are obtained by adding defining relations step by step and using a complicated simultaneous induction.

The following interesting properties of the free Burnside groups $B(r, n)$ for odd $n \geq 665$ and $r > 1$ were established in [4]. They are very close to the properties of absolutely free groups:

1) All finite subgroups and all commutative subgroups of $B(r, n)$ are cyclic.

2) The group $B(3, n)$ is isomorphically embedded in $B(2, n)$.

3) There exists an infinite descending chain of embedded normal subgroups

$$B(2, n) \subset G_1 \subset G_2 \subset \ldots \subset G_i \subset G_{i+1} \subset \ldots$$

4) The group $B(r, n)$ has an exponential growth and the growth function is very close to one of the absolutely free group with the same number of generators $r > 1$.

Later, in 1982, a nonamenability of the groups $B(r, n)$ for odd $n \geq 665$ and $r > 1$ was proved as well (von Neuman's problem). It is the only known a case when a nonamenable group satisfies a nontrivial identity. It was also proved that the symmetric random walk on these groups is transient (H. Kesten's problem).

There are various group theoretic problems that don't related to the periodic groups, but they also were solved by the author using the created method. We list some of them.

I. The first and very simple example of infinite irreducible system of group identities in two variables(1969) (a solution of the finite basis problem for group varieties, H. Neuman).

II. A construction of finitely generated noncommutative analogues of the additive group of rational numbers, i.e. the groups with an infinite intersection of any two nontrivial subgroups (1971). This groups $A(r,n)$ are central extensions of the group $B(r,n)$ by an infinite cyclic center generated by an element a. Adding to $A(r,n)$ one more relation $a^n = 1$ we get an example of finitely generated countable infinite group which admits only the discrete topology (A.A. Markov's problem).

III. New commutative and associative operations of periodic product of groups, satisfying the hereditary property relative to subgroups, 1976 (A.I. Maltsev's problem). The simplicity criterion for the periodic products of groups (proved in 1978) gave an opportunity to construct many new classes of finitely generated infinite periodic simple groups.

Among interesting results proved by another authors one should mention the work by A.Yu. Olshansky [6] who proposed a modification of Novikov–Adian method by using van Kampen diagrams instead of group transformation sequences. This modification works only for much larger values of exponents $n > 10^{10}$. He constructed the first examples of so called "Tarsky monsters", i.e. infinite periodic groups of prime exponent $n = p > 10^{75}$ with only proper subgroups of order p. Later S.I. Adian and I.G. Lysionok in the joint paper [8] constructed similar "Tarsky monsters" for any odd exponent $n \geq 1003$. We used the method in the original form as it was described in [4]. In these groups all proper subgroups are cyclic of orders dividing n.

S.V.Ivanov (1994) and I.G.Lysionok (1996) independently spread the Novikov–Adian theory to sufficiently large even exponents. They announced their results simultaneously in 1992 and finally proved the infiniteness of the groups $B(2,n)$ accordingly for $n = 512k \geq 2^48$ (Ivanov) and for $n = 16k \geq 8000$ (Lysionok). In fact both of them added into the simultaneous induction some new sentences that describe the finite subgroups of the considered Burnside groups.

Let me recall also one of various problems of Burnside type.

Working on the Burnside problem W. Magnus in 1950 formulated so called "restricted Burnside problem" (see [7]):

> *For any given pair $r, n \geq 2$ consider only finite periodic groups of the exponent n, generated by r generators. Does there exist a group of a maximal order for a fixed pair r, n?*

Of course, if the group $B(r, n)$ is finite then it is the maximal finite one. But as free Burnside groups $B(r, n)$ for large n are infinite this was a new problem, related to the finite groups. It seems more reasonable to call this problem *the Burnside-Magnus problem.* Magnus was the first who proposed an approach to investigate finite periodic groups using close connection between finite groups and Lie algebras with so called Engel condition. Many authors worked on this problem intended by W. Magnus. For instance E. Zelmanov received a Fields prize in 1994 for the positive solution of the restricted Burnside-Magnus problem.

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